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Transport of reactive solutes in unsteady annular flow subject to wall reactions

Suvadip Paul, B.S. Mazumder *

Fluvial Mechanics Laboratory, Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700 108, India

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ABSTRACT

The present paper concerns with the dispersion process in steady and oscillatory flows through an annular pipe in presence of reversible and irreversible reactions at the wall. Method of homogenization, a multiple-scale method of averaging, is adopted for deriving the effective transport equations. The main objective is to look into the effect of aspect ratio of the annular pipe on the dispersion coefficient due to the combined effect of axial convection and radial diffusion in steady and oscillatory flows along the annulus, subject to the kinetic reversible phase exchange and irreversible absorption at the outer wall. Results demonstrate that upto a certain critical value of aspect ratio, dispersion coefficient increases with increase of aspect ratio when the wall is retentive, though the wall inertness may lead to decrease of dispersion coefficient with increase of aspect ratio. The results would be useful to the medical practitioners working in the domain of catheterized artery.

1. Introduction

The study of longitudinal dispersion of tracers through an annular straight tube is of considerable interest due to its applications in the field of chemical, environmental and biomedical engineering. The first fundamental study on dispersion was that of Taylor [1] who showed that the dominant mechanism, whereby a scalar contaminant cloud (or solute) is spread in the steady laminar flow within straight tube, is dispersion – the interaction between non-uniform velocity and diffusion across the flow. Aris [2] extended Taylor's theory to include longitudinal diffusion and developed an approach 'method of moments' to analyze the asymptotic behaviour of second order moment about the mean.

An exact solution of the diffusion equation was obtained by Chatwin [3] to study the dispersion in oscillatory flow. Watson [4] used the concept proposed by Chatwin to study the passive contaminant dispersion in an oscillatory pressure-driven flow. Jimenez and Sullivan [5] used a probabilistic model to study the stream wise dispersion in unsteady laminar flow. Pedley and Kamm [6] studied the axial mass transport in an annular region in presence of an oscillatory flow field.

There exist a large number of studies on Taylor dispersion under the sole influence of irreversible reaction. Some of them considered channel flow (Mondal and Mazumder [7]) or tube flow (Jiang and Grotberg [8]) while in few cases annular tube flows (Sarkar and Jayaraman [9], Mazumder and Mondal [10]) were also considered. Dispersion coefficient affected by a reversible phase exchange has been studied by Davidson and Schroter [11]; Phillips and Kaye [12]. But very few studies includes the effect of both reversible and irreversible reactions. Though Purnama [13], Revelli and Ridolfi [14,15] and Ng [16,17] carried out this effect for flow through a tube or open-channel, but to the best of our knowledge no attempt has been made to examine dispersion phenomena through an annular tube considering both reversible and irreversible wall reactions.

Effect of aspect ratio on dispersion was studied by Mondal and Mazumder [18], Mazumder and Mondal [10], Sarkar and Jayaraman [9,19] and others. These studies reveal that aspect ratio of annular pipe has important contribution in dispersion process. But in all these cases, only the effect of irreversible reaction at the boundary is considered, though the model demands to have application in catheterized artery where the reversible reaction also plays important role.

The main objective of the present paper is to examine the influence of annularity on the transport process under the combined effects of reversible and irreversible wall reactions, when the flow is driven by a pressure gradient comprising of steady and periodic components. The inner wall of the outer tube is lined with a very thin layer made up of a retentive and reactive materials. The substance that undergoes Taylor dispersion in the fluid is subject to reversible phase exchange and irreversible absorption. These boundary reactions are the most important processes controlling the dispersion of solute in catheterized artery. A relatively fast rate of phase exchange and a relatively slow rate of absorption

* Corresponding author.
E-mail addresses: bsm46@yahoo.com, bijoy@isical.ac.in (B.S. Mazumder).

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were considered. Results are shown how the spreading of tracers is influenced by the aspect ratio, phase exchange parameter and absorption.

The insertion of catheter into an artery leads to a formation of an annular region between the catheter and the arterial wall (McDonald [20]). The fact that the lung and blood vessels have conductive walls (where phase exchange between the arterial wall and the flowing fluid (blood) and reaction with a reactant secreted by the wall tissue may take place) and the catheter, of course, acts as an impermeable boundary is reflected in our boundary conditions. The model will help us in understanding the indicator technique and other mechanisms in the branchial region. The insertion of a pipe with smaller diameter at the centerline of an artery brings the asymmetry to the flow, and the increase of aspect ratio leads to the existence of symmetry of the annular flow.

2. Velocity distribution

We consider a fully developed, axi-symmetric laminar flow of a homogeneous, incompressible viscous fluid through an annular pipe having inner radius b and outer radius a (i.e., $a > b$). We have used a cylindrical coordinate system in which the radial and axial co-ordinates are r and x respectively. The flow is assumed to be unidirectional and so the velocity has only axial component $u(r,t)$ which satisfies the Navier–Stokes equation as:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

where $\partial p/\partial x$ is the axial pressure gradient, ρ is the fluid density and ν is the kinematic viscosity.

The horizontal pressure gradient, which drives the flow, consists of steady and harmonically fluctuating components,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = P\left[1 + \psi\,\mathrm{Re}\left(e^{i\omega t}\right)\right]$$

where $P > 0$ is the steady part of the pressure gradient, ψ is a factor such that $P\psi$ is the amplitude of the oscillatory part of the pressure gradient.

The no-slip conditions on the surface of the inner and outer walls of the annular pipe i.e. $u(b,t) = 0$ and $u(a,t) = 0$, will produce the following velocity profile,

$$u(r,t) = u_s(r) + \mathrm{Re}\left[u_w(r)e^{i\omega t}\right] \tag{1}$$

where the steady component $(u_s(r))$ is given by,

$$u_s(r) = K\langle u_s\rangle\left[1 - \left(\frac{r}{a}\right)^2 - \frac{1-(b/a)^2}{\log(b/a)}\log\left(\frac{r}{a}\right)\right] \tag{2}$$

where $\langle u_s\rangle$ is the velocity averaged over the annular section given by

$$\langle u_s\rangle = \frac{Pa^2}{4\nu K} \quad \text{and} \quad K = \frac{2}{1+(b/a)^2+(1-(b/a)^2)/\log(b/a)}$$

and the unsteady component $(u_w(r))$ of the velocity is

$$u_w(r) = c_1 K_0\left(\frac{1-i}{\delta}r\right) + c_2 I_0\left(\frac{1-i}{\delta}r\right) - \frac{iP\psi}{\omega} \tag{3}$$

where

$$c_1 = \frac{iP\psi}{\omega}\left[\frac{I_0(\frac{1-i}{\delta}b) - I_0(\frac{1-i}{\delta}a)}{K_0(\frac{1-i}{\delta}a)I_0(\frac{1-i}{\delta}b) - K_0(\frac{1-i}{\delta}b)I_0(\frac{1-i}{\delta}a)}\right],$$

$$c_2 = \frac{iP\psi}{\omega}\left[\frac{K_0(\frac{1-i}{\delta}b) - K_0(\frac{1-i}{\delta}a)}{I_0(\frac{1-i}{\delta}a)K_0(\frac{1-i}{\delta}b) - I_0(\frac{1-i}{\delta}b)K_0(\frac{1-i}{\delta}a)}\right]$$

and

$$\delta = \sqrt{\frac{2\nu}{\omega}}.$$

Here I_0 and K_0 are respectively the modified Bessel function of first kind and second kind of order zero with imaginary arguments. The function I_0 and K_0 can be expressed as $I_0(ri^{1/2}) = \mathrm{ber}(r) + i\,\mathrm{bei}(r)$ and $K_0(ri^{1/2}) = \mathrm{ker}(r) + i\,\mathrm{kei}(r)$ where ber, bei, ker and kei are Kelvins functions of order zero.

For flow through a tube (i.e., when $b = 0$), the steady and unsteady components of the velocity are given by

$$u_s(r) = 2\langle u_s\rangle\left[1 - \left(\frac{r}{a}\right)^2\right] \quad \text{with } \langle u_s\rangle = \frac{Pa^2}{8\nu}$$

and

$$u_w(r) = -\frac{iP\psi}{\omega}\left[1 - \frac{J_0(\frac{1-i}{\delta}r)}{J_0(\frac{1-i}{\delta}a)}\right].$$

3. Governing equation and boundary conditions

Let us consider the transport of a chemical species through the annular gap of a tube. If the species is completely miscible with the fluid and $C(x,r,t)$ is the concentration (mass of species dissolved per bulk volume of the fluid) of the mobile phase, then C satisfies the mass transport equation as:

$$\frac{\partial C}{\partial t} + u(r,t)\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} + \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right), \quad b < r < a \tag{4}$$

along with the boundary conditions,

$$\frac{\partial C}{\partial r} = 0, \quad r = b, \tag{5}$$

$$-D\frac{\partial C}{\partial r} - \Gamma C = \frac{\partial C_s}{\partial t} = k(\alpha C - C_s), \quad r = a, \tag{6}$$

where D is the molecular diffusion coefficient assumed to be constant and $C_s(x,t)$ is the concentration (mass of species retained per unit surface area of the wall) of the immobile phase. Here Γ, k and α are the irreversible absorption rate, the reversible reaction rate and the partition coefficient respectively.

The boundary condition (5) states that there is no net transport of mass through the inner wall of the annular pipe. Last equality of the boundary condition (6) states that the rate of accumulation of the immobile phase in the outer wall is linearly proportional to the departure from local equilibrium between concentrations of the two phases on the outer wall of the annulus, and the first equality simply describes the irreversible reaction occurring at the surface of the outer wall.

4. Assumptions

The following assumptions are made for carrying out the perturbation analysis,

1. The length scale for the longitudinal spreading of the chemical cloud is much greater than the annular gap. It is meant that $x = O(L)$ and $r = O(a-b)$, where L is a characteristic longitudinal distance for the chemical transport. The ratio
$$\epsilon = (a-b)/L \ll 1 \tag{7}$$
is small enough to use as ordering parameter.
2. The oscillation period of the flow is so short that within this period there are no appreciable transport effects along the tube, though the effect of radial diffusion is not negligible. But the annular gap of the tube is so fine that diffusion across the entire annular section may be accomplished within this short time scale.
3. The two reactions are of different orders. The reversible phase exchange is much faster than the irreversible reaction. This ensures that local equilibrium can be largely achieved over a finite number of oscillations.

4. The rate of absorption is much slower than the rate of phase exchange and the former is comparable with the advection speed down the tube.
5. The Peclet number is equal to or greater than order of unity:

$$\mathrm{Pe} \equiv a\langle u_s\rangle/D \geqslant \mathrm{O}(1). \tag{8}$$

Under these assumptions, three distinct time scales may be defined as (Ng [16]),

$$T_0 = 2\pi/\omega = \mathrm{O}\big((a-b)^2/D\big) = \mathrm{O}\big(k^{-1}\big), \tag{9}$$

$$T_1 = L/\langle u_s\rangle = \mathrm{O}\big((a-b)\Gamma^{-1}\big) = T_0/\epsilon, \tag{10}$$

$$T_2 = L^2/D = T_0/\epsilon^2. \tag{11}$$

Based on these time scales, we may introduce accordingly

$$t_0 = t, \quad t_1 = \epsilon t, \quad t_2 = \epsilon^2 t \tag{12}$$

which are, respectively, the fast, medium and slow time variables.

5. Asymptotic analysis

The relative significance of the terms in the transport equation with the boundary conditions (4)–(6) are indicated below with the power of ϵ:

$$\frac{\partial C}{\partial t} + \epsilon u\frac{\partial C}{\partial x} = \epsilon^2 D\frac{\partial^2 C}{\partial x^2} + \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right), \quad b<r<a, \tag{13}$$

$$\frac{\partial C}{\partial r} = 0, \quad r=b, \tag{14}$$

$$-D\frac{\partial C}{\partial r} - \epsilon\Gamma C = \frac{\partial C_s}{\partial t} = k(\alpha C - C_s), \quad r=a. \tag{15}$$

Following the asymptotic expansion introduced by Fife and Nicholes [21], the concentrations C and C_s are expressed as,

$$C(x,r,t) = C^{(0)}(x,r,t_1,t_2) + \epsilon C^{(1)}(x,r,t_0,t_1,t_2) + \epsilon^2 C^{(2)}(x,r,t_0,t_1,t_2) + \mathrm{O}\big(\epsilon^3\big), \tag{16}$$

$$C_s(x,t) = C_s^{(0)}(x,t_1,t_2) + \epsilon C_s^{(1)}(x,t_0,t_1,t_2) + \epsilon^2 C_s^{(2)}(x,t_0,t_1,t_2) + \mathrm{O}\big(\epsilon^3\big) \tag{17}$$

and for the multiple-scale asymptotic analysis, the time derivative has been expanded as,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2}. \tag{18}$$

Here the terms involving a time scale T_1 or larger can be labeled as 'developed', while those terms that vanish exponentially as $t\to\infty$ with relaxation time proportional to T_0 are called 'transient' terms. Therefore, transient terms can be neglected for a time that is a few time larger than T_0. Since we are interested in a time scale equal to or larger than T_1, we may accordingly ignore all the transient effects. The expansions (16) and (17) contain only the so-called developed concentrations. The terms $C^{(n)}$ and $C_s^{(n)}$ ($n = 1,2,3,\ldots$) are purely oscillatory function of short time variable t_0 but the oscillatory effect does not show up on the zeroth order (i.e., $n=0$), and therefore the leading order concentrations are taken to be independent of the time variable t_0.

Using the expansions (16)–(18) in Eqs. (13)–(15) and equating the coefficients of like powers of ϵ from both sides, a system of differentials equations is obtained.

For zeroth order (O(1)) Eqs. (13)–(15) give

$$0 = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C^{(0)}}{\partial r}\right), \quad b<r<a, \tag{19}$$

$$\frac{\partial C^{(0)}}{\partial r} = 0, \quad r=b, \tag{20}$$

$$-D\frac{\partial C^{(0)}}{\partial r} = k\big(\alpha C^{(0)} - C_s^{(0)}\big), \quad r=a. \tag{21}$$

Eqs. (19) and (20) obviously imply that the leading order concentration is independent of r, i.e.,

$$C^{(0)} = C^{(0)}(x,t_1,t_2). \tag{22}$$

The boundary condition (21) then gives

$$C_s^{(0)} = \alpha C^{(0)}. \tag{23}$$

As expected, at the leading order the mobile phase of the chemical is at local equilibrium with the immobile phase.

For first order ($\mathrm{O}(\epsilon)$), Eqs. (13)–(15) give

$$\frac{\partial C^{(0)}}{\partial t_1} + \frac{\partial C^{(1)}}{\partial t_0} + u\frac{\partial C^{(0)}}{\partial x} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C^{(1)}}{\partial r}\right), \quad b<r<a, \tag{24}$$

$$\frac{\partial C^{(1)}}{\partial r} = 0, \quad r=b, \tag{25}$$

$$-D\frac{\partial C^{(1)}}{\partial r} - \Gamma C^{(0)} = \frac{\partial C_s^{(0)}}{\partial t_1} + \frac{\partial C_s^{(1)}}{\partial t_0} = k\big(\alpha C^{(1)} - C_s^{(1)}\big), \quad r=a. \tag{26}$$

Averaging Eqs. (24)–(26) w.r.t. the fast time variable t_0, we get

$$\frac{\partial C^{(0)}}{\partial t_1} + u_s\frac{\partial C^{(0)}}{\partial x} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{C}^{(1)}}{\partial r}\right), \quad b<r<a, \tag{27}$$

$$\frac{\partial \bar{C}^{(1)}}{\partial r} = 0, \quad r=b \tag{28}$$

and

$$-D\frac{\partial \bar{C}^{(1)}}{\partial r} - \Gamma C^{(0)} = \frac{\partial C_s^{(0)}}{\partial t_1} = k\big(\alpha\bar{C}^{(1)} - \bar{C}_s^{(1)}\big), \quad r=a, \tag{29}$$

where the overbar denotes time average over one period of oscillation and u_s is the steady velocity component. We further take cross-sectional average of (27) subject to the conditions (23), (28) and (29):

$$\frac{\partial C^{(0)}}{\partial t_1} + \frac{\langle u_s\rangle}{R}\frac{\partial C^{(0)}}{\partial x} + \frac{2a\Gamma}{R(a^2-b^2)}C^{(0)} = 0 \tag{30}$$

where

$$R = 1 + \frac{2a\alpha}{a^2-b^2} \tag{31}$$

is the retardation factor resulting from the reversible partitioning of the chemical into two phases. It is the ratio, when in equilibrium, of the total mass of the two chemical phases to the mass of the mobile phase per unit length of the annulus.

Using (23) in (26) and eliminating $\partial C^{(0)}/\partial t_1$ from (24)–(26) and (30), we rewrite Eqs. (24)–(26) as:

$$\frac{\partial C^{(1)}}{\partial t_0} + \left(u - \frac{\langle u_s\rangle}{R}\right)\frac{\partial C^{(0)}}{\partial x} - \frac{2a\Gamma}{R(a^2-b^2)}C^{(0)} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C^{(1)}}{\partial r}\right), \quad b<r<a, \tag{32}$$

$$\frac{\partial C^{(1)}}{\partial r} = 0, \quad r=b, \tag{33}$$

$$-D\frac{\partial C^{(1)}}{\partial r} = -\alpha\frac{\langle u_s\rangle}{R}\frac{\partial C^{(0)}}{\partial x} + \frac{\Gamma}{R}C^{(0)} + \frac{\partial C_s^{(1)}}{\partial t_0} = k\big(\alpha C^{(1)} - C_s^{(1)}\big) + \Gamma C^{(0)}, \quad r=a. \tag{34}$$

Eqs. (32)–(34) suggest that $C^{(1)}$ and $C_s^{(1)}$ are linearly proportional to $\partial C^{(0)}/\partial x$ and $C^{(0)}$. Accordingly, one can express these first order concentrations $C^{(1)}$ and $C_s^{(1)}$ into a sum of two terms: the first term is linearly proportional to $\partial C^{(0)}/\partial x$ with a coefficient containing steady and oscillatory components, while the second term is linearly proportional to $C^{(0)}$ with a steady coefficient:

$$C^{(1)} = \left[N(r) + \mathrm{Re}\left(B(r)e^{i\omega t_0}\right)\right]\frac{\partial C^{(0)}}{\partial x} + M(r)C^{(0)} \tag{35}$$

and

$$C_s^{(1)} = \left[N_s + \mathrm{Re}\left(B_s e^{i\omega t_0}\right)\right]\frac{\partial C^{(0)}}{\partial x} + M_s C^{(0)} \tag{36}$$

where the coefficients $N(r)$, N_s, $M(r)$, M_s, $B(r)$ and B_s satisfy the boundary value problems given below.

Substituting (35) and (36) into (32)–(34), and equating the steady term of the coefficient of $\partial C^{(0)}/\partial x$, we find the equation for $N(r)$ as:

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dN}{dr}\right) = u_s - \frac{\langle u_s\rangle}{R}, \quad b < r < a \tag{37}$$

with the boundary conditions

$$\frac{dN}{dr} = 0, \quad r = b, \tag{38}$$

$$-D\frac{dN}{dr} = -\alpha\frac{\langle u_s\rangle}{R} = k(\alpha N - N_s), \quad r = a. \tag{39}$$

Again equating the steady term associated with $C^{(0)}$, we get $M(r)$ and M_s:

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dM}{dr}\right) = -\frac{2a\Gamma}{R(a^2 - b^2)}, \quad b < r < a \tag{40}$$

with the boundary conditions

$$\frac{dM}{dr} = 0, \quad r = b, \tag{41}$$

$$-D\frac{dM}{dr} = \frac{\Gamma}{R} = k(\alpha M - M_s) + \Gamma, \quad r = a. \tag{42}$$

Similarly, we obtain the following equations for the complex functions $B(r)$ and B_s:

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dB}{dr}\right) = i\omega B + u_w, \quad b < r < a \tag{43}$$

with the boundary conditions,

$$\frac{dB}{dr} = 0, \quad r = b, \tag{44}$$

$$-D\frac{dB}{dr} = i\omega B_s = k(\alpha B - B_s), \quad r = a. \tag{45}$$

Introducing the dimensionless quantities

$$\hat{r} = \frac{r}{a}, \quad \lambda = \frac{b}{a}, \quad \hat{N} = \frac{N}{\langle u_s\rangle a^2/D}, \quad \hat{N}_s = \frac{N_s}{\langle u_s\rangle a^3/D},$$
$$\hat{M} = \frac{M}{\hat{\Gamma}}, \quad \hat{M}_s = \frac{M_s}{\hat{\Gamma}a},$$
$$D_a = \frac{ka^2}{D}, \quad \hat{\alpha} = \frac{\alpha}{a}, \quad \hat{\Gamma} = \frac{\Gamma a}{D}$$

the solutions (in dimensionless form) of Eqs. (37) and (40) subject to the prescribed boundary conditions are given by,

$$\begin{aligned}\hat{N}(\hat{r}) = {} & \hat{N}(\lambda) + \frac{K}{4}\left[\hat{r}^2 - \lambda^2 - \frac{1}{4}\left(\hat{r}^4 - \lambda^4\right)\right. \\ & \left. - \frac{1-\lambda^2}{\log(\lambda)}\left(\hat{r}^2\log(\lambda) - \lambda^2\log(\lambda) - \left(\hat{r}^2 - \lambda^2\right)\right)\right] \\ & - \frac{1}{4R}\left(\hat{r}^2 - \lambda^2\right) - \frac{K}{2}\left[\lambda^2 - \frac{\lambda^4}{2}\right. \\ & \left. - \frac{1-\lambda^2}{\log(\lambda)}\left(\lambda^2\log(\lambda) - \frac{\lambda^2}{2}\right)\right]\log\left(\frac{\hat{r}}{\lambda}\right) \\ & + \frac{1}{2R}\lambda^2\log\left(\frac{\hat{r}}{\lambda}\right),\end{aligned} \tag{46}$$

$$\hat{M}(\hat{r}) = \hat{M}(\lambda) - \frac{\hat{r}^2 - \lambda^2}{2(1-\lambda^2)R} + \frac{\lambda^2}{(1-\lambda^2)R}\log\left(\frac{\hat{r}}{\lambda}\right), \tag{47}$$

$$\hat{N}_s = \hat{\alpha}\hat{N}(1) + \frac{\hat{\alpha}}{D_a R} \tag{48}$$

and

$$\hat{M}_s = \hat{\alpha}\hat{M}(1) + \frac{2}{D_a(1-\lambda^2)R} \tag{49}$$

where $\hat{N}(\lambda)$ and $\hat{M}(\lambda)$ are undetermined constants unless a uniqueness condition is specified.

Here D_a, $\hat{\Gamma}$ and $\hat{\alpha}$ are respectively the Damkohler number (representing the kinetics of the phase exchange at the outer wall), absorption parameter (representing the rate of loss on the outer wall) and phase partition ratio (retention parameter).

The dimensionless form of the boundary value problem (43)–(45) is

$$\frac{1}{\hat{r}}\frac{d}{d\hat{r}}\left(\hat{r}\frac{d\hat{B}}{d\hat{r}}\right) + \hat{\eta}^2\hat{B} = \hat{u}_w, \quad \lambda < \hat{r} < 1, \tag{50}$$

$$\frac{d\hat{B}}{d\hat{r}} = 0, \quad \hat{r} = \lambda, \tag{51}$$

$$\frac{d\hat{B}}{d\hat{r}} = \hat{\eta}^2\hat{B}_s = -D_a(\hat{\alpha}\hat{B} - \hat{B}_s), \quad \hat{r} = 1, \tag{52}$$

where $\hat{B}$, $\hat{B}_s$, $\hat{u}_w$ and $\hat{\eta}$ are the dimensionless parameters defined as

$$\hat{B} = \frac{B}{\langle u_s\rangle a^2/D}, \quad \hat{B}_s = \frac{B_s}{\langle u_s\rangle a^3/D}, \quad \hat{u}_w = \frac{u_w}{\langle u_s\rangle}, \quad \hat{\eta} = a\eta,$$

with

$$\eta^2 = -\frac{2i\,Sc}{a^2\hat{\delta}^2}, \quad Sc = \frac{\nu}{D}, \quad \hat{\delta} = \frac{\delta}{a} = \frac{\sqrt{2}}{\Lambda}, \quad \Lambda = a\sqrt{\frac{\omega}{\nu}}.$$

Because of the complex structure of $\hat{u}_w(\hat{r})$ in Eq. (50), central difference scheme has been adopted to solve the equation under the prescribed boundary conditions.

Here Sc is the Schmidt number which is the ratio of viscous diffusion (ν) to the molecular diffusion (D) and $\hat{\delta}$ is the ratio of the Stokes boundary thickness ($\sqrt{2\nu/\omega}$) to the outer radius of the annular tube (a). It should be noted that $\hat{\delta}$ is inversely proportional to the dimensionless frequency parameter or Womersley number Λ which is the product of two dimensionless parameters, the Reynolds number ($\langle u_s\rangle a/\nu$) and the Strouhal number ($\omega a/\langle u_s\rangle$). The parameter Λ^2 is a measure of the ratio of the time (a^2/ν) required for viscosity to smooth out the transverse variation in vorticity to the period of oscillation ($1/\omega$). The number $\hat{\delta}$ increases with the oscillation period. Therefore, the higher the frequency, the smaller the value of $\hat{\delta}$. The Schmidt number Sc is of order unity when the fluid is a gas, and is much greater than unity when the fluid is a liquid. Keeping in mind the possible application of this model in catheterized artery, investigations are made taking $Sc = 10^3$. For dispersion to be significant, one may expect that, $\hat{\delta} \geqslant O(1)$ or the period of oscillation must be long enough for the Stokes boundary-layer thickness to be at least comparable with the annular gap of the tube.

For second order ($O(\epsilon^2)$), (13)–(15) give

$$\frac{\partial C^{(0)}}{\partial t_2}+\frac{\partial C^{(1)}}{\partial t_1}+\frac{\partial C^{(2)}}{\partial t_0}+u\frac{\partial C^{(1)}}{\partial x}=D\frac{\partial^2 C^{(0)}}{\partial x^2}+\frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C^{(2)}}{\partial r}\right),\quad b<r<a, \tag{53}$$

$$\frac{\partial C^{(2)}}{\partial r}=0,\quad r=b \tag{54}$$

and

$$-D\frac{\partial C^{(2)}}{\partial r}-\Gamma C^{(1)}=\frac{\partial C_s^{(0)}}{\partial t_2}+\frac{\partial C_s^{(1)}}{\partial t_1}+\frac{\partial C_s^{(2)}}{\partial t_0}=k\big(\alpha C^{(2)}-C_s^{(2)}\big),\quad r=a. \tag{55}$$

Averaging Eq. (53) w.r.t time followed by space subject to the boundary conditions (54) and (55) and using Eqs. (30), (35) and (36) in appropriate places one gets,

$$\frac{\partial C^{(0)}}{\partial t_2}+\zeta\frac{\langle u_s\rangle}{R}\frac{\partial C^{(0)}}{\partial x}+\chi\frac{2a\Gamma}{R(a^2-b^2)}C^{(0)}=\left[\frac{D}{R}+D_{Ts}+D_{Tw}\right]\frac{\partial^2 C^{(0)}}{\partial x^2} \tag{56}$$

with

$$\zeta=-\left\langle M\left(\frac{1}{R}-\frac{u_s}{\langle u_s\rangle}\right)\right\rangle-\frac{2aM_s}{R(a^2-b^2)}-\frac{2a\Gamma}{\langle u_s\rangle(a^2-b^2)}\left[\frac{\langle N\rangle}{R}+\frac{2aN_s}{R(a^2-b^2)}-N(a)\right], \tag{57}$$

$$\chi=-\frac{\langle M\rangle}{R}-\frac{2aM_s}{R(a^2-b^2)}+M(a), \tag{58}$$

$$D_{Ts}=\frac{1}{R}\left\langle N\left(\frac{\langle u_s\rangle}{R}-u_s\right)\right\rangle+\frac{2a\langle u_s\rangle N_s}{R^2(a^2-b^2)}, \tag{59}$$

and

$$D_{Tw}=-\frac{1}{2R}\operatorname{Re}\langle u_w B^*\rangle \tag{60}$$

where the asterisk denotes the complex conjugate.

Here ζ and χ are the higher order correction factors to the advection velocity and decay rate respectively. D_{Ts} and D_{Tw} are the respective dispersion coefficients due to steady and oscillatory part of the fluid motion and these are positive definite.

Introducing dimensionless quantities,

$$(\hat D_{Ts},\hat D_{Tw})=(D_{Ts},D_{Tw})/\big(\langle u_s\rangle^2a^2/D\big) \tag{60a}$$

and using $\hat N(\hat r)$, $\hat N_s$, $\hat M(\hat r)$ and $\hat M_s$ from Eqs. (46)–(49), expressions (57)–(59) can be written in dimensionless form as,

$$\begin{aligned}\zeta={}&-\hat\Gamma\big[-48+48R-8\log(\lambda)RD_a+42\lambda^6\log(\lambda)D_a\\&-18\lambda^2\log(\lambda)D_a-30\lambda^4\log(\lambda)D_a+27\lambda^6RD_a-46\lambda^6R\log(\lambda)D_a\\&+6\log(\lambda)D_a-63\lambda^4RD_a-18\lambda^6D_a+42\lambda^4D_a-30\lambda^2D_a\\&+36R\log(\lambda)\lambda^4D_a+18R\lambda^2\log(\lambda)D_a+6D_a-9RD_a+45R\lambda^2D_a\\&+24\log(\lambda)^2\lambda^6RD_a-48\log(\lambda)-48R\log(\lambda)\lambda^4+48\log(\lambda)R\\&-24\log(\lambda)^2\lambda^4D_a-24\log(\lambda)^2\lambda^6D_a+48\log(\lambda)\lambda^4+96\lambda^2\\&-48\lambda^4-96R\lambda^2+48\lambda^4R\big]/\big[12R^2D_a\big(\log(\lambda)-\lambda^2\log(\lambda)\lambda^4\\&-\log(\lambda)+\lambda^6\log(\lambda)+1-3\lambda^2+3\lambda^4-\lambda^6\big)\big],\end{aligned}$$

$$\begin{aligned}\chi={}&\big[-3D_a\lambda^4-8R+4D_a\lambda^2-8\lambda^2-D_a+8+8R\lambda^2\\&+4\lambda^4D_a\log(\lambda)\big]/4D_aR^2\big(-1+\lambda^2\big)^2,\end{aligned}$$

$$\begin{aligned}\hat D_{Ts}={}&\big[144-144R+102\log(\lambda)RD_a-288\lambda^6\log(\lambda)D_a\\&+144\lambda^2\log(\lambda)D_a-540\lambda^6RD_a-18D_a\log(\lambda)^2-54D_a\lambda^8\\&+708\lambda^6R\log(\lambda)D_a-36\log(\lambda)D_a+144R\lambda^6\log(\lambda)^2\\&-144R\log(\lambda)^2\lambda^2+144R\log(\lambda)^2\lambda^4-99D_a\lambda^8R^2\\&+162D_a\lambda^8R+648\lambda^4RD_a+180\lambda^6D_a-216\lambda^4D_a+108\lambda^2D_a\\&-372R\lambda^2\log(\lambda)D_a+72D_a\log(\lambda)^3\lambda^8+72D_a\log(\lambda)^3\lambda^4\\&-18D_a-144D_a\log(\lambda)^3\lambda^8R+36D_a\log(\lambda)^2\lambda^2\\&-144D_a\log(\lambda)^3\lambda^6R-219D_aR^2\log(\lambda)^2\lambda^8\\&+420D_a\lambda^8R\log(\lambda)^2+108D_a\lambda^4R^2\log(\lambda)^2\\&+108D_a\lambda^4R^2\log(\lambda)-144R\log(\lambda)^2+48D_aR\log(\lambda)^2\\&-33D_aR^2\log(\lambda)^2-198D_a\lambda^8\log(\lambda)^2-76D_aR^2\log(\lambda)\\&+144D_a\log(\lambda)^3\lambda^6+180D_a\lambda^8\log(\lambda)+72D_a\log(\lambda)^3\lambda^8R^2\\&-60D_aR\log(\lambda)^2\lambda^2-324D_aR\log(\lambda)^2\lambda^4-438D_a\lambda^8R\log(\lambda)\\&-464D_a\lambda^6R^2\log(\lambda)+184D_a\lambda^2R^2\log(\lambda)\\&+248D_a\lambda^8R^2\log(\lambda)+54RD_a+144D_a\log(\lambda)^2\lambda^6R^2\\&-324R\lambda^2D_a-144\lambda^4\log(\lambda)^2+144\lambda^2\log(\lambda)^2\\&-84\log(\lambda)^2\lambda^6RD_a-144\lambda^6\log(\lambda)^2+144\log(\lambda)^2\\&+288\log(\lambda)+288R\log(\lambda)\lambda^4-288\lambda^6R\log(\lambda)\\&+288R\lambda^2\log(\lambda)-45R^2D_a+288\lambda^6\log(\lambda)+234R^2\lambda^2D_a\\&-432R^2\lambda^4D_a+342R^2\lambda^6D_a-144\lambda^6-288\log(\lambda)R\\&+216\log(\lambda)^2\lambda^4D_a-36\log(\lambda)^2\lambda^6D_a-288\log(\lambda)\lambda^4\\&-432\lambda^2+432\lambda^4-288\lambda^2\log(\lambda)+432R\lambda^2-432\lambda^4R\\&+144\lambda^6R\big]/\big[144R^3\big(-\log(\lambda)^2-\lambda^2\log(\lambda)^2-2\log(\lambda)\\&+\lambda^4\log(\lambda)^2+2\log(\lambda)\lambda^4-1+3\lambda^2-3\lambda^4+2\lambda^2\log(\lambda)\\&+\lambda^6\log(\lambda)^2-2\lambda^6\log(\lambda)+\lambda^6\big)D_a\big].\end{aligned}$$

When $\lambda\to 0$, the dimensionless dispersion coefficient due to steady flow reduces to

$$\lim_{\lambda\to 0}\hat D_{Ts}=\frac{1}{R}\left(\frac{11}{48}-\frac{1}{3R}+\frac{1}{8R^2}\right)+\frac{2\hat\alpha}{R^3D_a}\quad\text{with } R=1+2\hat\alpha$$

which agrees with Ng [16].

The dispersion coefficient $\hat D_{Tw}$ due to oscillatory flow has been evaluated numerically from the expression (60a) through (60) with the known values of $\hat B(\hat r)$ and $\hat u_w(\hat r)$. Like steady component, oscillatory component of the dispersion coefficient $\hat D_{Tw}$ also agrees well with Ng [16] when $\lambda\to 0$.

It is mentioned here that two different normalization scales may be used for the dispersion coefficient due to oscillatory flow – (i) when the amplitude of the oscillatory pressure gradient is kept constant and the frequency is varied, in which a direct comparison between D_{Ts} and D_{Tw} can be possible, (ii) the flow can be under a volume-cycled oscillation such that the tidal or Stroke volume is maintained constant when the frequency is varied in oscillatory pumping.

For the second case, it is desirable to normalize D_{Tw} as:

$$\check D_{Tw}=D_{Tw}/\big(DV_T^2/a^6\big) \tag{60b}$$

where V_T $(=(2\pi a^2/\omega)|\langle u_w\rangle|)$ is the tidal volume that can be defined equal to twice the amplitude of the volume fluctuation (Watson [4]).

The undetermined constants $\hat N(\lambda)$ and $\hat M(\lambda)$ in the expressions $\hat N(\hat r)$ and $\hat M(\hat r)$ respectively have no effect on the evaluation of ζ, χ and $\hat D_{Ts}$. One can easily show that all terms involving the undetermined constants cancel out.

Eliminating $\partial C^{(0)}/\partial t_2$ from (53)–(56) and assuming the second order concentration of the form

$$C^{(2)} = \left[U(r) + \mathrm{Re}\left(X(r)e^{i\omega t_0}\right)\right]\frac{\partial C^{(0)}}{\partial x} + \left[V(r) + Y(r)\right]\frac{\partial^2 C^{(0)}}{\partial x^2} \tag{61}$$

and

$$C_s^{(2)} = \left[U_s + \mathrm{Re}\left(X_s e^{i\omega t_0}\right)\right]\frac{\partial C^{(0)}}{\partial x} + [V_s + Y_s]\frac{\partial^2 C^{(0)}}{\partial x^2} \tag{62}$$

where U and V are forced by the steady flow and X and Y by the oscillatory flow, we find the boundary value problems for U as:

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) = M\left(u_s - \frac{\langle u_s\rangle}{R}\right) - \zeta\frac{\langle u_s\rangle}{R} - \frac{2a\Gamma}{R(a^2-b^2)}N, \quad b<r<a \tag{63}$$

with the boundary conditions

$$\frac{dU}{dr} = 0, \quad r = b, \tag{64}$$

$$-D\frac{dU}{dr} - \Gamma N(r) = -(\alpha\zeta + M_s)\frac{\langle u_s\rangle}{R} - \frac{2a\Gamma}{R(a^2-b^2)}N_s = k[\alpha U - U_s], \quad r = a \tag{65}$$

and that for V is

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = N\left(u_s - \frac{\langle u_s\rangle}{R}\right) + D_{Ts}, \quad b<r<a \tag{66}$$

with the boundary conditions

$$\frac{dV}{dr} = 0, \quad r = b, \tag{67}$$

$$-D\frac{dV}{dr} = \alpha D_{Ts} - \frac{\langle u_s\rangle}{R}N_s = k[\alpha V - V_s], \quad r = a. \tag{68}$$

Similarly the function X satisfies the boundary value problem

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right) + i\omega X = -\frac{2a\Gamma}{(a^2-b^2)R}B + Mu_w, \quad b<r<a, \tag{69}$$

$$\frac{dX}{dr} = 0, \quad r = b, \tag{70}$$

$$-D\frac{dX}{dr} - \Gamma B = -\frac{2a\Gamma}{(a^2-b^2)R}B_s - i\omega X_s = k(\alpha X - X_s), \quad r = a \tag{71}$$

while Y satisfies

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{dY}{dr}\right) = \frac{1}{2}\mathrm{Re}(u_w B^*) + D_{Tw}, \quad b<r<a, \tag{72}$$

$$\frac{dY}{dr} = 0, \quad r = b, \tag{73}$$

$$-D\frac{dY}{dr} = \alpha D_{Tw} = k(\alpha Y - Y_s), \quad r = a. \tag{74}$$

Introducing dimensionless quantities

$$(\hat{U}, \hat{X}) = (U, X)/\left(\hat{\Gamma}\langle u_s\rangle a^2/D\right),$$
$$(\hat{U}_s, \hat{X}_s) = (U_s, X_s)/\left(\hat{\Gamma}\langle u_s\rangle a^3/D\right),$$
$$(\hat{V}, \hat{Y}) = (V, Y)/\left(\langle u_s\rangle^2 a^4/D^2\right),$$
$$(\hat{V}_s, \hat{Y}_s) = (V, Y)/\left(\langle u_s\rangle^2 a^5/D^2\right)$$

Eqs. (63), (66), (69) and (72) are solved subject to the boundary conditions.

Averaging the transport equation of third order w.r.t. time t_0 and space and using (35), (36) and (61), (62) in appropriate places and collecting the steady and oscillatory terms associated with $\partial^2 C^{(0)}/\partial x^2$, one gets the corresponding dispersion coefficients D_{Ts}^{Γ} and D_{Tw}^{Γ} due to steady and oscillatory part of the fluid motion as:

$$D_{Ts}^{\Gamma} = \frac{1}{R}\left\langle U\left(\frac{\langle u_s\rangle}{R} - u_s\right)\right\rangle + \frac{2a\langle u_s\rangle U_s}{R^2(a^2-b^2)} - \left(\langle M\rangle + \frac{2aM_s}{a^2-b^2}\right)\frac{D_{Ts}}{R} + \zeta\left(\langle N\rangle + \frac{2aN_s}{a^2-b^2}\right)\frac{\langle u_s\rangle}{R^2} + \left(\langle V\rangle - RV(a) + \frac{2aV_s}{a^2-b^2}\right)\frac{2a\Gamma}{R^2(a^2-b^2)} \tag{75}$$

and

$$D_{Tw}^{\Gamma} = -\frac{1}{2R}\mathrm{Re}\langle u_w X^*\rangle - \left(\langle M\rangle + \frac{2aM_s}{a^2-b^2}\right)\frac{D_{Tw}}{R} + \left(\langle Y\rangle - RY(a) + \frac{2aY_s}{a^2-b^2}\right)\frac{2a\Gamma}{(a^2-b^2)R}. \tag{76}$$

It may be mentioned here that D_{Ts}^{Γ} and D_{Ts}^{Γ} are the higher order corrections to D_{Ts} and D_{Ts} respectively due to wall absorption.

Finally, combining the first (Eq. (30)), second (Eq. (56)) and third order transport equations (not given in the text), we get the overall effective transport equation (using the time variable t):

$$\frac{\partial C^{(0)}}{\partial t} + (1+\zeta)\frac{\langle u_s\rangle}{R}\frac{\partial C^{(0)}}{\partial x} + (1+\chi)\frac{2a\Gamma}{R(a^2-b^2)}C^{(0)} = \left[D/R + D_{Ts} + D_{Tw} + D_{Ts}^{\Gamma} + D_{Tw}^{\Gamma}\right]\frac{\partial^2 C^{(0)}}{\partial x^2}. \tag{77}$$

The dimensionless dispersion coefficient $\hat{D}_{Ts}^{\Gamma}$ $(= \frac{D_{Ts}^{\Gamma}}{\langle u_s\rangle^2 a^2/D})$ can be evaluated from (75) on substituting the values of $\hat{U}$, $\hat{V}$, $\hat{U}_s$ and $\hat{V}_s$. Again one can show that the undetermined constants have no effects on $\hat{D}_{Ts}^{\Gamma}$. The expression for $\hat{D}_{Ts}^{\Gamma}$ is too lengthy to provide it in the text. We have employed the symbolic computer algebra package MAPLE to perform the tedious algebraic manipulations and computations.

In the limit when $\lambda \to 0$, the expression for $\hat{D}_{Ts}^{\Gamma}$, after making use of $R = 1 + 2\hat{\alpha}$, reduces to

$$\lim_{\lambda\to 0}\hat{D}_{Ts}^{\Gamma} = \frac{\hat{\Gamma}}{R^2}\left\{\frac{61}{240} - \frac{199}{288R} + \frac{5}{8R^2} - \frac{3}{16R^3} + \left[-\frac{11}{24} + \frac{2}{R} - \frac{3}{2R^2} - \frac{3}{RD_a}\left(1 - \frac{2}{R}\right)\right]\frac{4\hat{\alpha}}{RD_a}\right\}$$

which coincides with the expression obtained by Ng [16].

The dispersion coefficient $\hat{D}_{Tw}^{\Gamma}$ $(= D_{Tw}^{\Gamma}/(\langle u_s\rangle^2 a^2/D))$ given by (76) can be evaluated upon solving (69)–(74) for X, Y and Y_s. Again it may be shown that the undetermined constants have no effects on D_{Tw}^{Γ}. Since no closed-form analytical solutions are available for X and Y in general, it may be solved numerically by a second order central finite-difference scheme. But for the present study we have omitted the discussions on D_{Tw}^{Γ}. It is only a higher order correction to $\hat{D}_{Tw}$ due to irreversible wall absorption. In fact, our main emphasis is on the effect of aspect ratio.

It should be mentioned here that when the frequency of flow oscillation is sufficiently low, its period becomes comparable to the advection time-scale. This leads to a considerable simplified model. The steady and time varying component of the dispersion coefficient resulting from the oscillatory flow can be evaluated. Because of the complex form of $u_w(r)$ due to the aspect ratio, the analytical expressions of transport coefficients are very lengthy and are not presented here. The fast oscillation limit of the transport coefficients are also omitted for the interest of space.

6. Results and applications to catheterized artery

The study of mass transport has potential applications to pulmonary and cardiovascular flow of living systems (Grotberg [22]). This paper aims to develop a conception on how catheterization may influence the dispersion in physiological systems. With this in view the reversible and irreversible reactions are considered to occur simultaneously at the outer boundary (on the walls of blood vessel). The catheters are made of inert materials, where no reactions of the blood particles take place. A zero flux boundary condition is considered in this study for the inner boundary. The results of the present analysis are important in understanding the dispersion process through a catheterized artery with a reactive and retentive arterial wall. The tube of radius a can be considered as a blood vessel and the insertion of another tube of radius b (i.e., the catheter) into the blood vessel causes the formation of an annular region between the catheter and the arterial walls. Apart from the pulsatile nature of the blood flow, since lung and blood vessels involve conductive walls, to understand the indicator dilution technique and other mechanisms in the bronchial region, it is important to analyze the dispersion procedure considering the wall characteristics. The objective is to provide a correction for the catheter induced errors in the measured values based on the concept of longitudinal diffusion of tracer materials due to the combined action of the convection and diffusion. The parameter λ, the ratio of the catheter radius b to the arterial radius a, is varied from 0.1 to 0.8 to analyze the effect of the catheter size on dispersion phenomena. The catheterization increases the frictional resistance to flow through the artery and hence alters the flow field owing to changes of hemodynamic conditions in the artery. The result of this analysis may provide a correction for catheter insertion in the light of the behaviour of the dispersion coefficient. The fast and slow oscillation limits of the dispersion coefficients due to oscillatory flow may be of interest in the area of cardiovascular system. In the artificial ventilation of the lung (Grotberg [22]), the fast oscillation limits of the dispersion coefficients have significant importance. Slow and fast oscillation of the heart is directly related to the systolic and diastolic pressure of living-body.

In this paper an asymptotic analysis has been presented for the advection–diffusion transport of a chemical species in a flow through an annular tube, where the flow is generated using pulsatile pressure gradient with non-zero mean. The species is underwent linear reversible and irreversible reactions at the inner wall of the outer tube. Analytical expressions are derived for the various dispersion coefficients due to steady flow with retention and the combined effect of retention and absorption at the wall; whereas a numerical scheme has been adopted to compute the dispersion coefficient due to oscillatory flow. The behaviour of these components of the dispersion coefficient under the variation of various controlling parameters like aspect ratio λ, retention parameter $\hat{\alpha}$, oscillation parameter $\hat{\delta}$ (inverse of Womersley number Λ), Schmidt number Sc, Damkohler number D_a is described in this section. To relate the present study with arterial blood flow, the Schmidt number Sc is restricted to the order of 10^3. The main emphasis is given on how the components of dispersion coefficient change with the aspect ratio of the annular tube.

The behaviour of the steady component of normalized dispersion coefficient $\hat{D}_{Ts}$ with variation of aspect ratio λ is described in Fig. 1(a), (b) for different values of $\hat{\alpha}$ when $D_a = 1.0$. It is observed from the figures that, $\hat{D}_{Ts}$ shows its decrement [Fig. 1(a)] with the increase of aspect ratio when the wall is perfectly inert (i.e., $\hat{\alpha} = 0$), but for retentive wall ($\hat{\alpha} > 0$) the dependence of $\hat{D}_{Ts}$ on λ shows a little irregular behaviour [Fig. 1(b)]. Fig. 1(b) shows that for weak retention ($\hat{\alpha} = 0.05$, 0.10) $\hat{D}_{Ts}$ is growing with λ over the range 0.0–0.8 and then it suddenly drops in the narrow region of the annulus, but when the retention is strong ($\hat{\alpha} > 0.1$), $\hat{D}_{Ts}$ decreases with λ over the whole annular gap. So increase or decrease of steady component of the dispersion coefficient with λ depends upon the retention parameter $\hat{\alpha}$. For inert wall, the decrease of $\hat{D}_{Ts}$ with λ may be due to decrease of mean velocity with aspect ratio λ. It may be mentioned here that the magnitude of the steady component of dispersion is very less for perfectly inert wall compared with the retentive wall. It is interesting to note that the presence of catheter with increase in size inhibits the dispersion process when the wall is inert, whereas for retentive wall, increase of catheter size enhances the dispersion coefficient if the retention is weak, though strong retention in the wall may diminish the dispersion coefficient. Fig. 1(c) depicts the variation of $\hat{D}_{Ts}$ with $\hat{\alpha}$ for different values of the aspect ratio λ. It is observed from the figure that for a fixed aspect ratio λ, $\hat{D}_{Ts}$ increases for small values of $\hat{\alpha}$ within the range 0.2–0.3, reaches a maximum value and then it decreases with $\hat{\alpha}$, because they are much weighted down by the retardation factor R. With the increase of aspect ratio, the maximum value of $\hat{D}_{Ts}$ decreases. The deviation of $\hat{D}_{Ts}$ from its maximum value increases with the increase of λ i.e., more the value of λ, more sharp the decrease of $\hat{D}_{Ts}$.

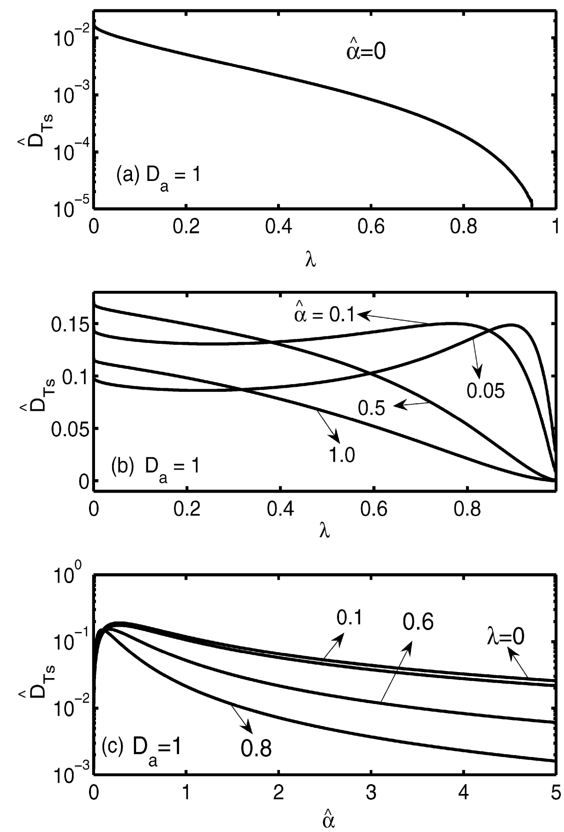

Fig. 1. Steady component of dispersion coefficient: (a) against the aspect ratio λ, when the wall is inert (b) against the aspect ratio λ, when the wall is retentive and (c) against the retention parameter $\hat{\alpha}$ for different values of the aspect ratio λ.

The variation of oscillatory component of the normalized dispersion coefficient $\hat{D}_{Tw}$ with λ for $Sc = 10^3, \hat{\delta} = 1$, $D_a = 1$ is plotted in Fig. 2(a) for different values of $\hat{\alpha}$. It is observed that $\hat{D}_{Tw}$ increases with the increase of aspect ratio over a wide range (0.0–0.85), reaches a maximum value and then within the remaining short range of λ it shows a sudden drop. So unless the aspect ratio of the tube exceeds a critical value, $\hat{D}_{Tw}$ increases as the annular gap of the tube decreases (increase of aspect ratio). The rate of increment of $\hat{D}_{Tw}$ with λ depends on the retention parameter $\hat{\alpha}$. The rate is high when the retention is weak. Also increase of $\hat{\alpha}$ leads to a decrease of $\hat{D}_{Tw}$ (Fig. 2(b)). It is observed from the figure that for all λ, $\hat{D}_{Tw}$ decreases monotonically with the increase of $\hat{\alpha}$. The effects of phase exchange on $\hat{D}_{Ts}$ and $\hat{D}_{Tw}$ are similar. Stronger kinetics of the phase exchange (i.e. smaller D_a) will give rise to a larger value of either dispersion coefficients (Ng [16]).

The dispersion coefficient $\hat{D}_{Tw}$ has been plotted in Fig. 3(a) against the aspect ratio λ for different values of $\hat{\delta}$ when $Sc = 10^3$, $D_a = 1$, $\hat{\alpha} = 0$. It is observed from the figure that for inert wall

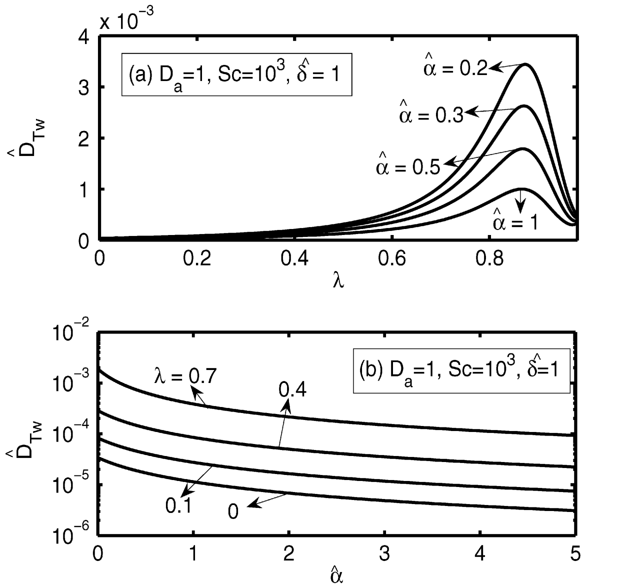

Fig. 2. Oscillatory component of the dispersion coefficient $\hat{D}_{Tw}$: (a) against the aspect ratio λ for different values of the partition coefficient $\hat{\alpha}$ and (b) against the retention parameter $\hat{\alpha}$ for different values of the aspect ratio λ.

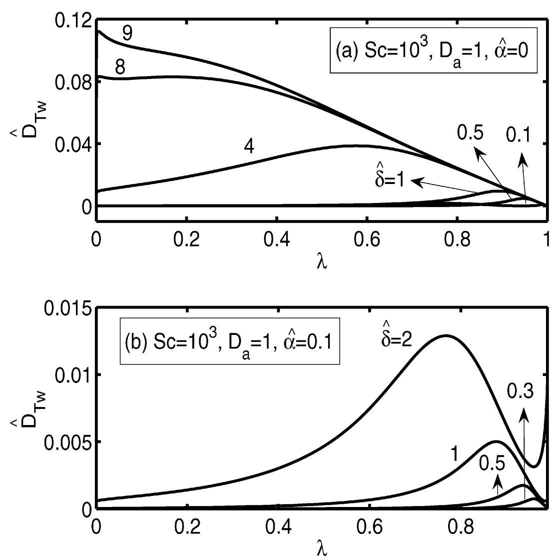

Fig. 3. Oscillatory component of the dispersion coefficient against the aspect ratio of the annular tube for different values of the oscillation parameter $\hat{\delta}$: (a) when the wall is inert and (b) when the wall is retentive.

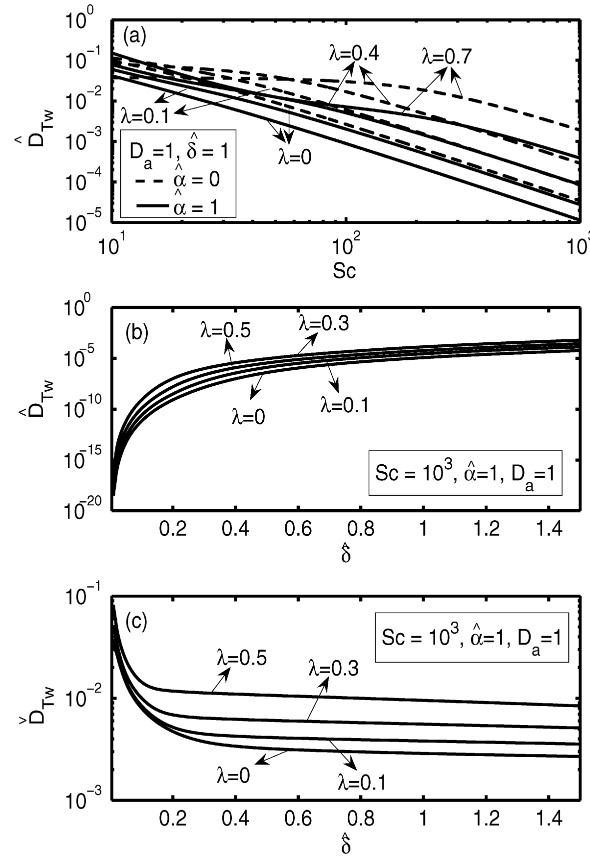

Fig. 4. Variation of oscillatory component of the dispersion coefficient for different values of the aspect ratio λ; against the Schmidt number Sc for $\hat{\delta}=1$, (b) dispersion coefficient for pressure gradient-cycled oscillation ($\hat{D}_{Tw}$) as functions of $\hat{\delta}$ when $Sc=10^3$ and $\hat{\alpha}=1$ and (c) the same as (b), but for dispersion coefficient due to volume-cycled oscillation ($\check{D}_{Tw}$).

($\hat{\alpha}=0$) and low oscillation parameter ($\hat{\delta}\approx 1$), $\hat{D}_{Tw}$ is practically ineffective with the variation of λ except for a narrow annular gap. Figure shows that for perfectly inert wall, $\hat{D}_{Tw}$ may increase with λ over a considerable range when the oscillation parameter $\hat{\delta}=4$. When $\hat{\delta}$ is sufficiently large, it is observed that $\hat{D}_{Tw}$ decreases with λ throughout its domain and it agrees well with the results of Sarkar and Jayaraman [9] and Mazumder and Mondal [10]. For fixed but small λ, $\hat{D}_{Tw}$ increases with increase of $\hat{\delta}$. As the annular gap becomes narrower, dispersion coefficient decreases with λ and apparently ineffective with the variation of $\hat{\delta}$. Fig. 3(b) shows the variation of $\hat{D}_{Tw}$ with λ for different values of the oscillation parameter $\hat{\delta}$ when the wall is retentive ($\hat{\alpha}>0$). It is clear from the figure that the increase of $\hat{\delta}$ leads to an increase of $\hat{D}_{Tw}$. This is because the decrease of $\hat{\delta}$ i.e., increase of frequency of oscillation results a reduction in the flow within the annular region, hence the reduction of dispersion coefficient. The influence of oscillation parameter $\hat{\delta}$, when it is small, is not that effective on dispersion coefficient for $0<\lambda<0.4$ (Fig. 3(b)). However large oscillation parameter $\hat{\delta}$ has appreciable contribution on the oscillatory component of the dispersion coefficient for all λ. The behaviour of $\hat{D}_{Tw}$ with the variation of λ is found to be consistent with the results obtained from Fig. 3(b) even when the wall is inert with small $\hat{\delta}$ or when the wall is retentive with large $\hat{\delta}$. It is observed that like steady component, oscillatory component of the dispersion coefficient is very less in case of non-retentive wall compared to that when the wall is retentive.

The variation of $\hat{D}_{Tw}$ with Schmidt number Sc is shown in Fig. 4(a) for different values of $\hat{\alpha}$ and λ. It is seen that higher value of Sc leads to smaller $\hat{D}_{Tw}$ for retentive ($\hat{\alpha}=1$) as well as inert ($\hat{\alpha}=0$) wall. As we discussed earlier, based on different normalization scales, there exists two dispersion coefficients: $\hat{D}_{Tw}$ (given by (60a)) and $\check{D}_{Tw}$ (given by (60b)). The former is for a pressure gradient-cycled oscillation, while the latter is for a volume-cycled oscillation. As shown in Fig. 4(b), (c), the oscillation frequency have qualitatively different effects on these two normalized dispersion coefficients. While $\hat{D}_{Tw}$ increases monotonically with $\hat{\delta}$ (i.e., the increase of Womersley number Λ inhabits the dispersion process (Mazumder and Mondal [10])), the opposite is true for $\check{D}_{Tw}$. Therefore under a volume-cycled oscillation, a larger dispersion coefficient will result from a higher Womersley number, because a stronger pressure force needs to be applied to maintain a constant tidal volume. From Fig. 4(b), it is seen that $\hat{D}_{Tw}$ asymptotically reaches a steady state after a certain $\hat{\delta}$ and the value of $\hat{\delta}$ depends on $\hat{\alpha}$ and λ. For a fixed retention parameter $\hat{\alpha}$, $\hat{D}_{Tw}$ approaches to stationary state earlier if λ is large.

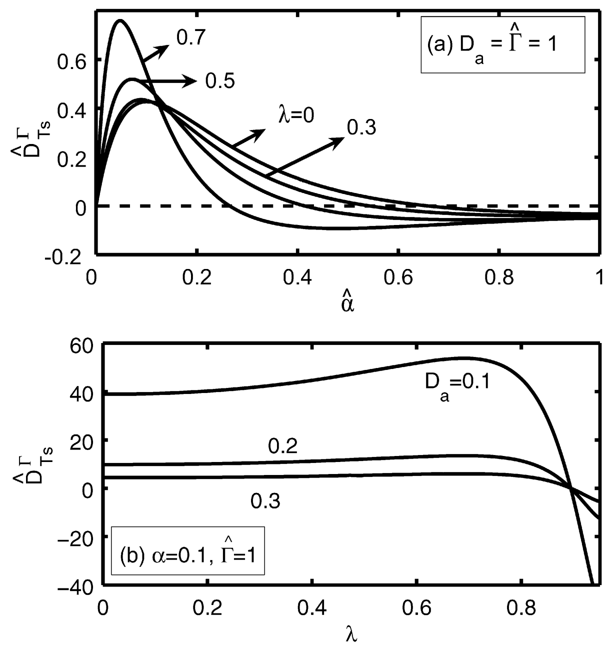

Fig. 5. (a) Variation of dispersion coefficient $\hat{D}^{\Gamma}_{Ts}$ against $\hat{\alpha}$ for different values of λ when $D_a = 1$, $\hat{\Gamma} = 1$ and (b) variation of $\hat{D}^{\Gamma}_{Ts}$ against λ for different values of the kinetics of the phase exchange D_a.

The steady component of normalized dispersion coefficient $\hat{D}^{\Gamma}_{Ts}$ due to the combined effect of wall retention and absorption has been plotted in Fig. 5(a) against the retention parameter $\hat{\alpha}$ for various values of aspect ratio λ when $D_a = 1$ and in Fig. 5(b) against λ for various values of reversible reaction rate parameter D_a. Figure shows that mild retention in the wall results a sharp increase in $\hat{D}^{\Gamma}_{Ts}$. It is clear that for small values of $\hat{\alpha}$ and $D_a = 1$, the peak of $\hat{D}^{\Gamma}_{Ts}$ increases with the increase in λ. $\hat{D}^{\Gamma}_{Ts}$ first shows an increment with $\hat{\alpha}$, then it decreases upto a certain value and then it remains almost independent of $\hat{\alpha}$. This phenomenon becomes more significant with the increase in λ. From Fig. 5(b), $\hat{D}^{\Gamma}_{Ts}$ shows a slight increase with λ over the range 0.0–0.8 and within the remaining short range it decreases sharply. With the decrease of D_a, the effect of λ on $\hat{D}^{\Gamma}_{Ts}$ appears to be more significant. It is observed that $\hat{D}^{\Gamma}_{Ts}$ decreases with increase of D_a over the whole range of λ except at the narrow gap of the annular tube, where $\hat{D}^{\Gamma}_{Ts}$ increases.

7. Conclusions

The stream-wise dispersion of tracer materials released in an oscillatory flow through an annular pipe (termed as catheterized artery) with reversible and irreversible reactions at the outer wall has been studied using the method of homogenization. The present results are compared with the existing results available in the literature when the aspect ratio approaches to zero and are found in good agreement. The following important conclusions that can be drawn from the present study:

1. The steady component of $\hat{D}_{Ts}$ decreases with increase of λ for perfectly inert wall, whereas for retentive wall ($\hat{\alpha} > 0$), depending upon the degree of retention, it may increase with λ.
2. For non-retentive wall ($\hat{\alpha} = 0$) and low oscillation parameter $\hat{\delta}$, $\hat{D}_{Tw}$ is practically ineffective with the insertion of catheter except in a narrow gap, whereas for sufficiently large $\hat{\delta}$ (low Womersley number), $\hat{D}_{Tw}$ decreases with λ throughout its domain.
3. Unless the aspect ratio of the annular tube exceeds a certain critical value, the oscillatory component of the dispersion coefficient $\hat{D}_{Tw}$ increases with decrease of annular gap (increase of λ). The rate of increment of $\hat{D}_{Tw}$ with λ depends on the retention parameter and the rate is high when the retention is weak.
4. The frequency of flow oscillation shows qualitatively different effects on two types of normalized dispersion coefficients resulting from the pressure gradient-cycled oscillation and volume-cycled oscillation.
5. With the possible application in catheterized artery, the results show that the presence of catheter with increase in size inhibits the dispersion process when the wall is non-retentive, whereas for retentive wall, the insertion of catheter enhances the dispersion if the retention is weak, though strong retention in the wall reduces the dispersion coefficient.

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